June 14 (R) M.A. Kprime2

1. Solve the equation

$$5 \tanh x + 7 = 5 \operatorname{sech} x$$

Give each answer in the form ln k where k is a rational number.

(5)

1. $5\tanh x + 7 = 5\operatorname{sech} x$

$\therefore \dfrac{5\sinh x}{\cosh x} - \dfrac{5}{\cosh x} + 7 = 0$

$\times \cosh x$

$\therefore 5\sinh x + 7\cosh x - 5 = 0$

$\therefore \frac{5}{2}e^{x} - \frac{5}{2}e^{-x} + \frac{7}{2}e^{x} + \frac{7}{2}e^{-x} - 5 = 0$

$\therefore 6e^{x} + e^{-x} - 5 = 0$

$\times e^{x} \Rightarrow 6e^{2x} - 5e^{x} + 1 = 0$

$(2e^{x} - 1)(3e^{x} - 1) = 0$

$\Rightarrow e^{x} = \frac{1}{2} \quad \therefore x = \ln \frac{1}{2}$

$e^{x} = \frac{1}{3} \quad \therefore x = \ln \frac{1}{3}$

2.

$$9x^2 + 6x + 5 \equiv a(x + b)^2 + c$$

(a) Find the values of the constants a, b and c.

(3)

Hence, or otherwise, find

(b) $\displaystyle\int \frac{1}{9x^2 + 6x + 5}\,dx$

(2)

(c) $\displaystyle\int \frac{1}{\sqrt{9x^2 + 6x + 5}}\,dx$

(2)

2(a). $9x^2 + 6x + 5 = 9\left(x^2 + \frac{2}{3}x + \frac{5}{9}\right)$

$= 9\left[\left(x + \frac{1}{3}\right)^2 + \frac{4}{9}\right]$

$= 9\left(x + \frac{1}{3}\right)^2 + 4$

(b) $\displaystyle\frac{1}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2 + \frac{4}{9}}\,dx = \frac{1}{9} \times \frac{3}{2} \arctan\left(\frac{x + \frac{1}{3}}{2/3}\right) + C$

$= \frac{1}{6} \arctan\left(\frac{3x+1}{2}\right) + C$

(c) $\displaystyle\frac{1}{3}\int \frac{1}{\sqrt{\left(x+\frac{1}{3}\right)^2 + \frac{4}{9}}}\,dx = \frac{1}{3} \operatorname{arsinh}\left(\frac{3x+1}{2}\right) + C$

3. The curve C has equation

$$y = \frac{1}{2}\ln(\coth x), \quad x > 0$$

(a) Show that

$$\frac{dy}{dx} = -\operatorname{cosech} 2x$$

(3)

The points A and B lie on C.

The x coordinates of A and B are $\ln 2$ and $\ln 3$ respectively.

(b) Find the length of the arc AB, giving your answer in the form $p \ln q$, where p and q are rational numbers.

(6)

3(a) $y = \frac{1}{2}\ln(\coth x)$

$\therefore \frac{\partial y}{\partial x} = \frac{1}{2} \times \frac{\frac{\partial}{\partial x}(\coth x)}{\coth x}$

$\cancel{c^2+s}$

$\frac{c^2}{s^2} - \frac{s^2}{s^2} = \frac{1}{s^2}$

$= \frac{1}{2} \times \frac{-\operatorname{cosech}^2 x}{\coth x}$

$\coth^2 - 1 = \operatorname{cosech}^2$

$= \frac{1}{2} \times -\frac{\frac{1}{\sinh^2 x}}{\frac{\cosh x}{\sinh x}} \times \frac{\cancel{\sinh^2 x}}{\sinh^2 x}$

use: $\frac{1}{2}\sinh 2x = \sinh x \cosh x$

$= \frac{1}{2} \times -\frac{1}{\cosh x \sinh x}$

$= \frac{1}{2} \times -\frac{1}{\frac{1}{2}\sinh 2x} = -\operatorname{cosech} 2x$

as required.

(b) $y = \frac{1}{2}\ln(\coth x) \Rightarrow \frac{\partial y}{\partial x} = -\operatorname{cosech} 2x$

$$\therefore \left(\frac{\partial y}{\partial x}\right)^2 = \operatorname{cosech}^2 2x$$

$$\frac{C^2}{S^2} - \frac{S^2}{S^2} = \frac{1}{S^2}$$

$$\coth^2 - 1 = \operatorname{cosech}^2$$

$$\therefore S = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\, \partial x$$

$$\therefore S = \int_{\ln 2}^{\ln 3} \sqrt{1 + \operatorname{cosech}^2 2x}\, \partial x$$

Use : $\operatorname{cosech}^2 2x + 1 = \coth^2 2x$

$$\therefore S = \int_{\ln 2}^{\ln 3} \coth 2x \, \partial x = \left[\frac{1}{2}\ln|\sinh 2x|\right]_{\ln 2}^{\ln 3}$$

$\ln|\sinh x|$

$\frac{\cosh x}{\sinh x}$

$$= \frac{1}{2}\ln|\sinh \ln 9| - \frac{1}{2}\ln|\sinh \ln 4|$$

$$= \frac{1}{2}\ln\left|\frac{9 - 9^{-1}}{2}\right| - \frac{1}{2}\ln\left|\frac{4 - 4^{-1}}{2}\right|$$

$$= \ln\left(\frac{40}{9}\right) - \ln\left(\frac{15}{8}\right) = \ln\left(\frac{64}{27}\right) / 2$$

$$= \frac{1}{2}\ln\left(\frac{64}{27}\right) \quad = \cancel{\ln\left[\left(\frac{4}{3}\right)^3\right] = 3\ln\left(\frac{4}{3}\right)}$$

$$\cancel{= 3\ln 4 - 3\ln 3 = \ln\left[\left(\frac{3}{4}\right)^{-3}\right] =}$$

4.

$$I_n = \int_0^{\sqrt{3}} (3 - x^2)^n \,dx, \quad n \geqslant 0$$

(a) Show that, for $n \geqslant 1$

$$I_n = \frac{6n}{2n+1} I_{n-1}$$

(6)

(b) Hence find the exact value of I_4, giving your answer in the form $k\sqrt{3}$ where k is a rational number to be found.

(5)

4(a). $I_n = \int_0^{\sqrt{3}} (3-x^2)^n \, dx$

$\therefore$ Let $u = (3-x^2)^n \qquad u' = n(3-x^2)^{n-1} \times -2x$

$= -2nx(3-x^2)^{n-1}$

$v' = 1 \qquad v = x$

$\therefore \cancel{I_n = -2nx(3-x^2)^{n-1}}$

$\therefore I_n = \left[x(3-x^2)^n\right]_0^{\sqrt{3}} + 2n \int_0^{\sqrt{3}} x^2 (3-x^2)^{n-1} \, dx$

$\therefore I_n = 2n \int_0^{\sqrt{3}} x^2 (3-x^2)^{n-1} \, dx$

Use $x^2 \equiv 3 - (3-x^2)$

$\therefore I_n = 2n \int_0^{\sqrt{3}} 3(3-x^2)^{n-1} - (3-x^2)^n \, dx$

$\therefore I_n = 2n (3I_{n-1} - I_n)$

$\therefore I_n = 6n I_{n-1} - 2n I_n$

$\therefore \quad (2n+1)I_n = 6n\, I_{n-1}$

$\Rightarrow \quad I_n = \dfrac{6n}{2n+1} I_{n-1}$

as required.

(b) $I_4 = \dfrac{8}{3} I_3 = \dfrac{8}{3}\left(\dfrac{18}{7} I_2\right) = \dfrac{48}{7}\left(\dfrac{12}{5} I_1\right)$

$= \dfrac{576}{35}\left(2\, I_0\right)$

$= \dfrac{1152}{35}\displaystyle\int_0^{\sqrt{3}} (3-x^2)^0\, dx$

$\cancel{= \dfrac{1152}{35}\left[3x - \dfrac{1}{3}x^3\right]_0^{\sqrt{3}}}$ $= \dfrac{1152}{35}\left[x\right]_0^{\sqrt{3}}$

$\cancel{= \dfrac{1152}{35}\left(3\sqrt{3} - \sqrt{3} - 0\right)}$ $= \dfrac{1152}{35}\sqrt{3}$

$\cancel{\therefore I_4 = \dfrac{2304}{35}\sqrt{3}}$ $\therefore I_4 = \dfrac{1152}{35}\sqrt{3}$

5. The ellipse E has equation

$$x^2 + 9y^2 = 9$$

The point $P(a\cos\theta, b\sin\theta)$ is a general point on the ellipse E.

(a) Write down the value of a and the value of b. **(1)**

The line L is a tangent to E at the point P.

(b) Show that an equation of the line L is given by

$$3y\sin\theta + x\cos\theta = 3$$ **(3)**

The line L meets the x-axis at the point Q and meets the y-axis at the point R.

(c) Show that the area of the triangle OQR, where O is the origin, is given by

$$k\operatorname{cosec} 2\theta$$

where k is a constant to be found. **(3)**

The point M is the midpoint of QR.

(d) Find a cartesian equation of the locus of M, giving your answer in the form $y^2 = f(x)$. **(4)**

5(a). $\frac{x^2}{9} + y^2 = 1$

$\therefore \quad a = 3 \quad , \quad b = 1$

Question 5 continued

(b) $\cancel{\frac{dy}{d\theta} = \frac{-3\sin\theta}{\cos\theta}}$

at P, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta}{-3\sin\theta}$

$y - y_1 = m(x - x_1)$

$\therefore \cancel{y - b\sin\theta = -\frac{1}{3}\frac{\cos\theta}{\sin\theta}(x - a\cos\theta)}$

$\therefore \cancel{y - b\sin\theta = \frac{-\cos\theta}{3\sin\theta}x +}$

$y - \sin\theta = \frac{-\cos\theta}{3\sin\theta}(x - 3\cos\theta)$

$\therefore \quad y - \sin\theta = -\frac{\cos\theta}{3\sin\theta}x + \frac{\cos^2\theta}{\sin\theta}$

$\times 3\sin\theta \Rightarrow 3y\sin\theta - 3\sin^2\theta = -\cos\theta\, x + 3\cos^2\theta$

$\therefore \quad 3y\sin\theta + x\cos\theta = 3(\cos^2\theta + \sin^2\theta)$
$= 3$

$\therefore \quad 3y\sin\theta + x\cos\theta = 1$ as required.

(c) @ Q, $y=0$

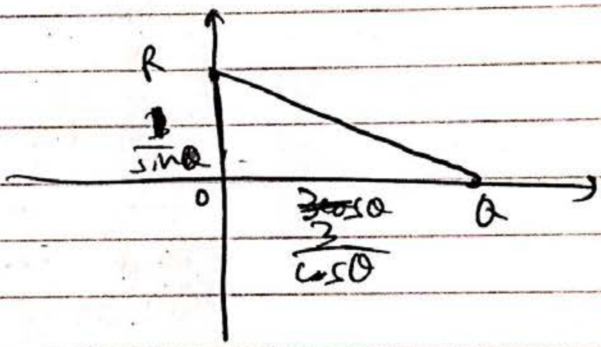
$\therefore \; x\cos\theta = 3 \Rightarrow x = \dfrac{3}{\cos\theta}$

$Q\left(\dfrac{3}{\cos\theta}, 0\right)$

@ R, $x=0 \Rightarrow 3y\sin\theta = 3$

$\therefore \; y = \dfrac{1}{\sin\theta}$

$\therefore \; R\left(0, \dfrac{1}{\sin\theta}\right)$



$\therefore \text{Area} = \frac{1}{2}|OQ| \times |OR|$

$= \frac{1}{2} \times \dfrac{3}{\cos\theta} \times \dfrac{1}{\sin\theta}$

$= \dfrac{3}{2} \times \dfrac{1}{\sin\theta\cos\theta} = \dfrac{3}{2} \times \dfrac{1}{\frac{1}{2}\sin 2\theta}$

$\therefore = \dfrac{3}{2} \times \dfrac{2}{\sin 2\theta} = 3\,\text{cosec}\,2\theta$

~~k = 3~~

$k=3$

Q5

(Total 11 marks)

(d) M: $\left(\dfrac{3}{2\cos\theta}, \dfrac{1}{2\sin\theta}\right)$

$\therefore x = \dfrac{3}{2\cos\theta} \Rightarrow \cos\theta = \dfrac{3}{2x}$

$y = \dfrac{1}{2\sin\theta} \Rightarrow \sin\theta = \dfrac{1}{2y}$

$\therefore \sin^2\theta + \cos^2\theta = 1 = \dfrac{1}{4y^2} + \dfrac{9}{4x^2}$

$\therefore 1 - \dfrac{9}{4x^2} = \dfrac{1}{4y^2}$

$\therefore \dfrac{4x^2-9}{4x^2} = \dfrac{1}{4y^2}$

$\therefore 4y^2 = \dfrac{4x^2}{4x^2-9}$

$\therefore \cancel{y^2 = \dfrac{9x^2}{4x^2-9}} \Rightarrow y^2 = \dfrac{x^2}{4x^2-9}$

$16x^2-36$

$\dfrac{9^2}{4x^2-9}$

Q6

(Total 11 marks)

6. The symmetric matrix $\mathbf{M}$ has eigenvectors $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

with eigenvalues 5, 2 and -1 respectively.

(a) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$

(4)

Given that $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$

(b) show that

$$\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

(2)

(c) Hence find the matrix $\mathbf{M}$.

(5)

Find Normalize E. vectors

6(a) $\underset{\sim}{x}_1 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$\underset{\sim}{x}_2 = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$\underset{\sim}{x}_3 = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$\therefore \quad P = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}$ //

$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ //

(b) $P^{-1} = P^{T}$

$\cancel{P^T M}$ $P^{T} M P = D$

$\therefore P^{-1} M P = D$

$\times P$ $\quad P P^{-1} M P = P D$

$\therefore M P = P D$

$\cancel{x}$ $M P P^{-1} = P D P^{-1} \Leftarrow \times P^{-1}$

$\therefore M = P D P^{-1}$ as required.

(c) $P^{T} = P^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$

$\cancel{\therefore M =}$ $M = P D P^{-1}$

$PD = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 10 & -4 & -1 \\ 10 & 2 & 2 \\ 5 & 4 & -2 \end{pmatrix}$

$$\therefore M = \left(\frac{1}{3}\right)^2 \begin{pmatrix} 10 & -4 & -1 \\ 10 & 2 & 2 \\ 5 & 4 & -2 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$$

$$\therefore M = \frac{1}{9} \begin{pmatrix} 27 & 18 & 0 \\ 18 & 18 & 18 \\ 0 & 18 & 9 \end{pmatrix}$$

$$\therefore M = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

7. The curve C has equation

$$y = e^{-x}, \qquad x \in \mathbb{R}$$

The part of the curve C between $x = 0$ and $x = \ln 3$ is rotated through 2π radians about the x-axis.

(a) Show that the area S of the curved surface generated is given by

$$S = 2\pi\int_0^{\ln 3} e^{-x}\sqrt{1 + e^{-2x}}\,dx$$

(3)

(b) Use the substitution $e^{-x} = \sinh u$ to show that

$$S = 2\pi\int_{\text{arsinh}\,\alpha}^{\text{arsinh}\,\beta} \cosh^2 u\,du$$

where α and β are constants to be determined.

(5)

(c) Show that

$$2\int \cosh^2 u\,du = \frac{1}{2}\sinh 2u + u + k$$

where k is an arbitrary constant.

(2)

(d) Hence find the value of S, giving your answer to 3 decimal places.

(2)

7(a). $y = e^{-x} \Rightarrow \dfrac{dy}{dx} = -e^{-x} \Rightarrow \left(\dfrac{dy}{dx}\right)^2 = e^{-2x}$

Limits are from $x=0$ to $x = \ln 3$

$\therefore \quad S = 2\pi \displaystyle\int_0^{\ln 3} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$\Rightarrow \quad S = 2\pi \displaystyle\int_0^{\ln 3} e^{-x}\sqrt{1 + e^{-2x}}\,dx$

as required.

Question 7 continued

$\frac{1}{e^x} = \sinh u$

(b) $e^{-x} = \sinh u \Rightarrow$

$\therefore \frac{dx}{du} \cdot -e^{-x} = \cosh u$

$\therefore -\frac{dx}{du} \frac{1}{e^x} = \cosh u$

$e^x = \frac{1}{\sinh u}$

$\therefore dx = -e^x \cosh u \, du$

$\therefore dx = -\frac{\cosh u}{\sinh u} du$

$e^{-x}\sqrt{1+e^{-2x}} \equiv \sinh u \sqrt{1+\sinh^2 u} = \sinh u \sqrt{\cosh^2 u}$

$\therefore e^{-x}\sqrt{1+e^{-2x}} \equiv \sinh u \cosh u$

Consider limits: from $x=0$ to $x = \ln 3$

$\therefore \cancel{\int}$

$x=0$: $x=0 \Rightarrow e^{-0} = \sinh u$

$\therefore 1 = \sinh u \Rightarrow u = \text{arsinh}(1)$

$x = \ln 3$: $x = \ln 3 \Rightarrow e^{-\ln 3} = \sinh u$

$\therefore \frac{1}{3} = \sinh u \Rightarrow u = \text{arsinh}\left(\frac{1}{3}\right)$

$\therefore$ New limits are from $u = \text{arsinh}(1)$

to $u = \text{arsinh}\left(\frac{1}{3}\right)$

Now:

$dx = -\frac{\cosh u}{\sinh u}\, du$

& $e^{-x}\sqrt{1+e^{-2x}} \equiv \sinh u \cosh u$

$\therefore S = 2\pi \int_0^{\ln 3} e^{-x}\sqrt{1+e^{-2x}}\, dx = 2\pi \int_{\text{arsinh}\, 1}^{\text{arsinh}\frac{1}{3}} \sinh u \cosh u \cdot -\frac{\cosh u}{\sinh u} du$

$\therefore S = -2\pi \int_{\text{arsinh}\, 1}^{\text{arsinh}\frac{1}{3}} \cosh^2 u\, du = 2\pi \int_{\text{arsinh}\frac{1}{3}}^{\text{arsinh}\, 1} \cosh^2 u\, du$

as required

$\beta = 1 \qquad \alpha = \frac{1}{3}$

(C) $\cosh^2 u + \sinh^2 u = \cosh 2u$

$\therefore 2\cosh^2 u - 1 = \cosh 2u$

$\therefore \cosh^2 u = \frac{1}{2}(\cosh 2u + 1)$

$\therefore 2\int \cosh^2 u\, du = \left(2 \times \frac{1}{2}\right) \int \cosh 2u + 1\, du$

$= \cancel{\left[\ \frac{1}{2}\right.} = \frac{1}{2}\sinh 2u + u + k$

as required.

$$S = 2\pi \int_{\text{arsinh}\frac{1}{3}}^{\text{arsinh} 1} \cosh^2 u \, du = \pi\left[\frac{1}{2}\sinh 2u + u\right]_{\text{arsinh}\frac{1}{3}}^{\text{arsinh} 1}$$

$\text{arsinh } 1 = \ln(1+\sqrt{2})$

$\text{arsinh } \frac{1}{3} = \ln\left(\frac{1+\sqrt{10}}{3}\right)$

$$= \pi\left(\frac{1}{2}\sinh\left[2\ln(1+\sqrt{2})\right] + \ln(1+\sqrt{2}) - \frac{1}{2}\sinh\left[2\ln\frac{1+\sqrt{10}}{3}\right] - \ln\frac{1+\sqrt{10}}{3}\right)$$

$$= \pi\left(1.414\ldots + \ln(1+\sqrt{2}) - 0.3513\ldots - \ln\frac{1+\sqrt{10}}{3}\right)$$

$$= \pi \times 1.616\ldots$$

$$= 5.0\cancel{7}9 \text{ } (3\cancel{d}p)$$

Q7

(Total 12 marks)

Turn over

8. The plane Π_1 has vector equation $\mathbf{r}.\begin{pmatrix}2\\1\\3\end{pmatrix} = 5$

The plane Π_2 has vector equation $\mathbf{r}.\begin{pmatrix}-1\\2\\4\end{pmatrix} = 7$

(a) Find a vector equation for the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a scalar parameter.

(6)

The plane Π_3 has cartesian equation

$$x - y + 2z = 31$$

(b) Using your answer to part (a), or otherwise, find the coordinates of the point of intersection of the planes Π_1, Π_2 and Π_3

(3)

8) a) $\Pi_1 : \quad 2x + y + 3z = 5$

$\Pi_2 : \quad -x + 2y + 4z = 7$

Solve simultaneously:

$\Rightarrow \quad x = 2y + 4z - 7$

$\therefore \quad 4y + 8z - 14 + y + 3z = 5$

$\therefore \quad 5y + 11z = 19$

$\therefore \quad y = \frac{19}{5} - \frac{11}{5}z$

$\therefore \quad x = \frac{38}{5} - \frac{22}{5}z + 4z - 7$

$\therefore \quad x = \frac{3}{5} - \frac{2}{5}z$

$\therefore \ x = \frac{3}{5} - \frac{2}{5}z$

$y = \frac{19}{5} - \frac{11}{5}z$,

$z = z$ Let $z = \lambda$

$\Rightarrow \underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/5 \\ 19/5 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2/5 \\ -11/5 \\ 1 \end{pmatrix}$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 3/5 \\ 19/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2/5 \\ -11/5 \\ 1 \end{pmatrix}$ $\cancel{\frac{1}{5}}$

λ is arbitrary.

or $\underset{\sim}{r} = \begin{pmatrix} 3/5 \\ 19/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -11 \\ 5 \end{pmatrix}$

(b)

~~x − 2y +~~ $x - y + 2z = 31$

$\therefore \ \frac{3}{5} - \frac{2}{5}\lambda - \frac{19}{5} + \frac{11}{5}\lambda + 2\lambda = 31$

$\therefore \ \frac{19}{5}\lambda = \frac{171}{5}$

$\Rightarrow \lambda = 9$

$\Rightarrow$ @ intersection $\underset{\sim}{r} = \begin{pmatrix} -3 \\ -16 \\ 9 \end{pmatrix}$